# CHIRAL PHASE TRANSITIONS

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September 4, 2001

**Abstract**

We show that melting the quark mass, the scalar $\sigma$ mass and the quark condensate leads uniquely to the quark-level SU(2) linear $\sigma$ model field theory. Upon thermalization, the chiral phase transition curve requires $T_c = 2f_\pi^{CL} \approx 180\ MeV$ when $\mu = 0$, while the critical chemical potential is $\mu_c = m_q \approx 325\ MeV$. Transition to the superconductive phase occurs at $T_c^{(SC)} = \Delta/\pi e^{-\gamma_E}$. Coloured diquarks suggest $T_c^{(SC)} < 180\ MeV$.

Invited talk at the Chiral Fluctuations in Hadronic Matter International Workshop, Orsay, France, Sept. 2001

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# 1 Introduction

According to the now prevailing views, it is expected that at high temperature and/or density the standard hadronic matter will undergo a phase transition to a new state. Such prospects have motivated present experimental searches for the signatures of the quark-gluon plasma in high-energy heavy ion collisions. The appearance of such state(s) may also affect properties of strongly interacting matter at such extreme conditions as those occurring inside of neutron stars. Clearly, if we are to understand the underlying phenomena better, various theoretical studies should be pursued.

Temperature dependence of the properties of low density QCD has been explored fairly well. The basis for our understanding is provided by lattice field theory calculations which, for QCD with two massive flavours, exhibit a phase transition to a chirally symmetric phase at $T_c = 173 \pm 8\ MeV$ for improved staggered fermions [1]. Dependence on baryon charge density or chemical potential $\mu$ is much more hypothetical, since lattice Monte Carlo techniques are not well suited to the case when $\mu$ is non-zero. Nonetheless, on the basis of general arguments and of an insight gained from various models, one may conjecture the structure of the phase diagram in QCD with two massless quarks.

Thus, the salient features of this QCD phase diagram in the $T\mu$ plane are known or expected as follows (see Fig.1 [2]).
i) When temperature $T$ is raised at $\mu = 0$, the chiral QCD condensate $\langle \bar{q}q \rangle$ melts at $T_c \approx 173 \pm 8\ MeV$ or so, and the system enters into a quark-gluon plasma (QGP) phase, where the chiral condensate is identically zero. The transition from the low temperature hadronic phase to the high temperature QGP phase is most probably of second order. Because the QGP and hadronic phases differ by the expectation value of $\langle \bar{q}q \rangle$, a (curved) line of phase transitions must begin at $T = T_c$, $\mu = 0$ and continue for $\mu > 0$ without terminating. By continuity for low $\mu$, these phase transitions must be of second order (this is indicated by a dotted line in Fig. 1).

ii) When the chemical potential $\mu$ of the system is increased at zero temperature, one first encounters a transition from the phase of zero number density to that of nuclear matter density. This transition is of first order and by continuity remains such also at low $T$. In Fig. 1 this is indicated by a solid line. The hadronic gas at low $T$ and small $\mu$ has number density $n(T,\mu) \neq 0$, and is connected analytically to the nuclear matter phase at high enough $T_0$. The latter is related to the binding energy of nucleons in nuclear matter ($T_0 \approx 10\ MeV$).
iii) When the chemical potential and number density are further increased at zero temperature, the interactions of quarks become weak due to (QCD) asymptotic freedom and colour interactions become screened at shorter and shorter lengths. Thus, nonperturbative phenomena such as chiral symmetry breaking should be absent at sufficiently high $\mu$ and a phase transition should occur at some $\mu_c$. Most likely, it is of first order. Since regions to the left and right of the point $\mu = \mu_c$, $T = 0$ differ in the value of the quark condensate, a line of first order transitions (marked solid) must start at this point and continue away from $T = 0$ without terminating. Most likely, it joins the line of second order phase transitions emanating from $\mu = 0$, $T = T_c$ at a tricritical point $T_3$.
iv) There are arguments that in the region of low $T$ to the right of $\mu_c$, one should expect a nonvanishing value of a diquark condensate $\langle qq \rangle$ which spontaneously breaks colour symmetry. In this region, for temperatures below $T_c^{(SC)}$ (of order 100 MeV), one expects to find colour superconductivity. The nature of this phase transition to a non-superconducting phase at higher temperatures is not clear yet and, consequently, this transition line is marked with dashes.

In the following it will be shown how some of the above features as well as other aspects of the underlying theory are predicted in a linear sigma model with a built-in dynamical symmetry breaking structure.

## 2 Chiral phase transition

The word "chiral" means handedness in Greek. Defining left- and right- handed charges $2Q^i_\pm = Q^i \pm Q^i_5$, the SU(2) current (charge) algebra of the 1960s reduces to $[Q^i_\pm, Q^j_\pm] = i\epsilon_{ijk}Q^k_\pm$, $[Q^i_\pm, Q^j_\mp] = 0$. We consider chiral low energy field theories of the linear $\sigma$ model ($L\sigma M$) [3, 4], four-fermion [5] and infrared QCD forms. When a field theory involving fermions (quarks) is thermalized [6], the fermion propagator is replaced by

$$\frac{i(\not p + m)}{p^2 - m^2} - \frac{2\pi\delta(p^2 - m^2)(\not p + m)}{e^{|p_0/T|} + 1}. \tag{1}$$

We shall show in many ways that the chiral symmetry restoration temperature $T_c$ ("melting" constituent quarks, scalar $\sigma$ mesons, or the quark condensate) is for 2 flavours in the chiral limit (CL)

$$T_c = 2f_\pi^{CL} \approx 180\ MeV. \tag{2}$$

Here the scale $f_\pi^{CL}$ is fixed from experiment [7] to be 90 MeV. This $T_c$ scale is compatible with computer lattice results [1] $T_c = 173 \pm 8$ MeV.

i) First we melt the SU(2) quark tadpole graph [8]:

$$0 \leftarrow m_q(T_c) = m_q + \frac{8N_c g^2 m_q}{-m_\sigma^2}\left(\frac{T_c^2}{2\pi^2}\right) J_+(0), \tag{3}$$

with quark-meson coupling $g$, colour number $N_c$ and $J_+(0) = \int_0^\infty xdx(e^x + 1)^{-1} = \pi^2/12$. Cancelling out the $m_q$ scale, Eq(3) gives (assuming $N_c = 3$)

$$T_c^2 = \frac{m_\sigma^2}{g^2}. \tag{4}$$

ii) Next we melt the scalar $\sigma$ mass with coupling $\lambda\Phi_\sigma^4/4$, to find [9]

$$0 \leftarrow m_\sigma^2(T_c) = m_\sigma^2 - 6\lambda\left(\frac{T_c^2}{2\pi^2}\right) J_-(0), \tag{5}$$

with $J_-(0) = \int_0^\infty xdx(e^x - 1)^{-1} = \pi^2/6$, resulting in

$$T_c^2 = 2m_\sigma^2/\lambda. \tag{6}$$

Taking the $L\sigma M$ tree-level relation $\lambda = m_\sigma^2/2f_\pi^2$ we note that the $m_\sigma$ mass scale divides out of Eq.(6) and recovers [9] Eq.(2) $T_c = 2f_\pi$. Also, the ratio of Eq.(4) to Eq.(6) gives

$$\lambda = 2g^2, \tag{7}$$

independent of the $m_\sigma$ and $T_c$ scales. Moreover, Eq.(7) holds at tree and at loop level as we shall show in Sec.3. Then with $T_c^2 = 4f_\pi^2$ from Eqs(2,6), melting the quark tadpole in Eq.(4) in turn requires

$$g^2 T_c^2 = 4f_\pi^2 g^2 = m_\sigma^2, \tag{8}$$

which implies the $L\sigma M$ [3, 4] quark-level Goldberger-Treiman relation (GTR) $f_\pi g = m_q$ and the famous NJL relation [5] $m_\sigma = 2m_q$. We shall return to this combined $L\sigma M$-NJL scheme [3, 4] in Sect.3.

iii) Finally we melt the quark condensate $\langle \bar{q}q(T_c)\rangle$ in QCD:

$$0 \leftarrow \langle \bar{q}q(T_c)\rangle_{m_q} = \langle \bar{q}q\rangle + 2 \cdot 4N_c m_q \left(\frac{T_c^2}{2\pi^2}\right) J_+(0), \tag{9}$$

in analogy with Eq.(3). Here the factor of 2 is due to thermalizing both $q$ and $\bar{q}$ in $\langle \bar{q}q\rangle$ (as opposed to the flavour factor of 2 in Eq.(3) due to $u$ and $d$ quarks). At a 1 GeV scale, the QCD condensate is known to have the value $\langle -\bar{q}q\rangle_{1\ \mathrm{GeV}} \approx (250\ \mathrm{MeV})^3$ for QCD coupling [10] $\alpha_s \approx 0.5$. But on the quark mass shell (where $m_q \approx M_N/3 \approx 315$ MeV) this condensate runs down to [11]

$$\langle -\bar{q}q\rangle_{m_q} = 3m_q^3/\pi^2 \approx (215\ \mathrm{MeV})^3. \tag{10}$$

In between, this condensate (10) freezes out at [12] $\alpha_s \approx 0.75$ near the $600 - 700$ MeV $\sigma$ mass. In any case, applying Eq.(10) to the quark condensate melting condition Eq.(9) gives for $J_+(0) = \pi^2/12$,

$$T_c^2 = 3m_q^2/\pi^2 \quad \text{or} \quad T_c = 2[m_q/(2\pi/\sqrt{3})]. \tag{11}$$

Thus Eq.(11) again requires $T_c = 2f_\pi^{CL}$, but only if the meson-quark QCD coupling is frozen out at the value

$$g = 2\pi/\sqrt{3} = 3.6276. \tag{12}$$

This infrared QCD relation Eq.(12) was earlier stressed in ref.[13]. Its numerical value can be tested by the GTR ratio of CL masses $g = (M_N/3)/f_\pi^{CL} \approx 315/90 \approx 3.5$.

Moreover the QCD effective $\alpha_s$ version of Eq.(12) is [11]

$$\alpha_s^{eff} = C_{2F}\alpha_s(m_\sigma) = (4/3)(\pi/4) = \pi/3, \tag{13}$$

which also follows from Eq.(2) or $g^2/4\pi = \pi/3$, the latter being the $L\sigma M$ version as we shall see in Sec.3. Of course $\alpha_s^{eff} = \pi/3 \approx 1$ is where the coupling is large and the quarks condense. Another theoretical interpretation of Eq.(12) is as a $Z = 0$ compositeness condition for the $L\sigma M$ [14]. This justifies that an elementary scalar $\sigma(600)$ is only slightly less in mass than the bound state $\bar{q}q$ vector $\rho(769)$ i.e. the UV chiral cutoff must be about 750 MeV separating elementary particles from bound states (in the SU(2) $L\sigma M$ field theory).

# 3 Dynamically generated SU(2) $L\sigma M$ at $T = 0$

Given the above thermal analysis which indirectly finds the T-independent relations $f_\pi^{CL} \approx 90$ MeV, $f_\pi g = m_q \approx 325$ MeV, $\lambda = 2g^2$, $g = 2\pi/\sqrt{3}$, $m_\sigma = 2m_q \approx 650$ MeV, we first note that the latter $\sigma$ mass is presumably $f_0(400-1200)$ as listed in the 1996,1998, 2000 PDG tables [15]. Accordingly, we now turn our attention to the SU(2) $L\sigma M$ field theory at $T = 0$.

The original interacting part of the SU(2) $L\sigma M$ lagrangian is [3]

$$\mathcal{L}_{L\sigma M}^{int} = g\bar{\psi}(\sigma + i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi + g'\sigma(\sigma^2 + \boldsymbol{\pi}^2) - \lambda(\sigma^2 + \boldsymbol{\pi}^2)^2/4, \tag{14}$$

$$g = m_q/f_\pi \quad , \quad g' = m_\sigma^2/2f_\pi = \lambda f_\pi. \tag{15}$$

Refs. [3] work only in tree order in a spontaneous symmetry breaking (SSB) context, and do not specify $g$, $g'$, $\lambda$ except that they obey the chiral relation

Eqs.(15). However, the dynamical symmetry breaking (DSB) version in loop order invoking the nonperturbative Nambu-type gap equations $\delta f_\pi = f_\pi$ and $\delta m_q = m_q$ was worked out in refs. [4] giving $g = 2\pi/\sqrt{3}$, $g' = 2gm_q$, $\lambda = 8\pi^2/3 \approx 26.3$. These gap equations are respectively (for $đ^4p \equiv d^4p/(2\pi)^4$)

$$1 = -i4N_c g^2 \int (p^2 - m_q^2)^{-2} đ^4 p \tag{16}$$

$$1 = 8iN_c g^2/(-m_\sigma^2) \int (p^2 - m^2)^{-1} đ^4 p, \tag{17}$$

where the log-divergent gap Eq.(16) (LDGE) follows from the quark loop version of $\langle 0|A_\mu^3|\pi^0\rangle = if_\pi q_\mu$ and also is valid in the usual NJL model. Note that the UV cutoff in the LDGE Eq.(16) is $\Lambda \approx 2.3m_q \approx 750$ MeV, consistently distinguishing the elementary $\pi(140)$, $\sigma(650)$ from the bound states $\rho(770)$, $\omega(780)$, $a_1(1260)$ as anticipated in Sec.2 [4]. The quadratic divergent Eq.(17) is solved by using the cutoff-independent dim.reg. lemma [4] for $2l = 4$:

$$\int đ^4 p \left[\frac{m_q^2}{(p^2 - m_q^2)^2} - \frac{1}{p^2 - m_q^2}\right] = \lim \frac{i m_q^{2l-2}}{(4\pi)^2}[\Gamma(2-l) + \Gamma(1-l)] = \frac{-i m_q^2}{(4\pi)^2} \tag{18}$$

because $\Gamma(2-l)+\Gamma(1-l)] \to -1$ as $l \to 2$ due to the gamma function identity $\Gamma(z+1) = z\Gamma(z)$. Alternatively Eq.(18) follows by starting with the partial fraction identity

$$\frac{m^2}{(p^2 - m^2)^2} - \frac{1}{p^2 - m^2} = \frac{1}{p^2}\left[\frac{m^4}{(p^2 - m^2)^2} - 1\right], \tag{19}$$

and integrating over $đ^4p$ while neglecting the massless tadpole $\int đ^4 p/p^2 = 0$ (as is also done in dim. reg., analytic, zeta function, and Pauli-Villars regularizations [4]). Then the right-hand side of Eq.(18) immediately follows, so Eq.(18) is more general than dim. reg.

Combining Eq.(18) with Eq.(17) and using the LDGE Eq.(16) gives

$$m_\sigma^2 = 2m_q^2(1 + g^2 N_c/4\pi^2), \tag{20}$$

which when combined with Eq.(12) implies $m_\sigma^2/2m_q^2 = 1 + 1 = 2$, the famous NJL relation. Note that 1+1=2 is not a delicate partial cancellation. Instead the loop order $L\sigma M$ neatly extends the tree order $L\sigma M$ relations Eqs.(15) to

$$m_\sigma = 2m_q, \qquad \text{when} \quad g = 2\pi/\sqrt{N_c}. \tag{21}$$

In fact the $\sigma$ mass computed from bubble plus tadpole graphs gives [4]

$$m_\sigma^2 = 16iN_c g^2 \int đ^4p \left[ \frac{m_q^2}{(p^2-m_q^2)^2} - \frac{1}{(p^2-m_q^2)} \right] = \frac{N_c g^2 m_q^2}{\pi^2} \tag{22}$$

using the dim. reg. lemma Eq.(18). Then applying Eq.(12) reduces Eq.(22) to the NJL relation Eq.(21). Further, the LDGE Eq.(16) nonperturbatively "shrinks" the $u$, $d$ quark triangle for $g_{\sigma\pi\pi}$ and the quark box for $g_{\sigma\sigma\pi\pi}$, $g_{\pi\pi\pi\pi}$ to [4]

$$g_{\sigma\pi\pi} = -8ig^3 N_c m_q \int (p^2-m_q^2)^{-2} đ^4p = 2gm_q = g' \tag{23}$$

$$\lambda_{box} = -8iN_c g^4 \int (p^2-m_q^2)^{-2} đ^4p = 2g^2 = g'/f_\pi = \lambda_{tree}. \tag{24}$$

Then $g_{\sigma\pi\pi} = g'$ when the GTR and also the NJL relation are valid. Also note that $\lambda_{box} = \lambda_{tree} = 2g^2$ in Eq.(24) was one of our major conclusions of Sec.2 in Eq.(7).

In effect, we have employed the general colour number $N_c$ when fitting physical data, but on occasion (Eqs.(10-27)) we have used $N_c = 3$ (phenomenologically based on the $\pi^0 \to 2\gamma$ decay rate). Also a theoretical basis for $N_c = 3$ is due to B. W. Lee's null tadpole condition [16]. Specifically the true (DSB) vanishing vacuum [not the false (SSB) nonvanishing vacuum] is characterized by the null sum of SU(2) $L\sigma M$ quark and meson tadpoles [16]. This leads to the CL equation [4]

$$\langle\sigma\rangle = 0 = -i8N_c g m_q \int (p^2-m_q^2)^{-1} đ^4p + 3ig' \int (p^2-m_\sigma^2)^{-1} đ^4p. \tag{25}$$

Using the $L\sigma M$ relations $g = m_q/f_\pi$, $g' = m_\sigma^2/2f_\pi$, multiplying through by $f_\pi$ and invoking dimensional analysis to replace the respective quadratic divergent integrals by $m_q^2$, $m_\sigma^2$ we see that Eq.(25) requires [4]

$$N_c(2m_q)^4 = 3m_\sigma^2, \tag{26}$$

where the factor of 3 is due to $\sigma - \sigma - \sigma$ combinatorics. But since we know this $L\sigma M$-NJL scheme demands $2m_q = m_\sigma$ (as in Eqs.(21,23), this null tadpole condition Eq.(26) demands [4] $N_c = 3$. There are alternative schemes [17] that suggest $N_c \to \infty$, but they are not grounded on our SU(2) $L\sigma M$-NJL couplings which require $N_c = 3$ when $m_\sigma = 2m_q$ in Eq.(26).

# 4 Colour superconductivity

Returning to the thermalization version (of the $L\sigma M$) in Sec.2, we extend this analysis to include a thermal chemical potential in order to comment on the $\mu \approx \mu_c$ region of Fig. 1. Following the first reference in ref.[9] we use exponential statistical mechanics factors $[e^{(E\pm\mu)/T} + 1]^{-1}$ for fermions. Then, melting the quark condensate gives a generalization of Eq.(11):

$$T_c^2 + \frac{3\mu_c^2}{\pi^2} = \frac{3m_q^2}{\pi^2}. \tag{27}$$

Dividing Eq.(27) above by $3m_q^2/\pi^2$ and using Eq.(11) to form the GTR, Eq.(27) becomes

$$\frac{T_c^2}{(2f_\pi)^2} + \frac{\mu_c^2}{m_q^2} = 1, \tag{28}$$

a (chiral) ellipse in the $T - \mu$ plane. From Eq.(28) by putting $\mu_c = 0$ or $T_c = 0$ one can read off both the critical temperature for the case of a vanishing chemical potential (yielding $T_c = 2f_\pi \approx 180\ MeV$), or critical chemical potential for a vanishing temperature (leading to $\mu_c = m_q \approx 325\ MeV$).

Stated another way, we recall that the low temperature ($T_c \approx 2^o\ K$) condensed matter BCS equation [18] can be expressed analytically as

$$2\Delta/T_c = 2\pi e^{-\gamma_E} \approx 2 \cdot 1.764 \approx 3.528, \tag{29}$$

where the Euler constant is $\gamma_E = 0.5772157$. A link to particle physics is replacing the condensed matter energy gap $\Delta$ by the (constituent) quark mass $m_q \approx 325\ MeV$. Then the above BCS equation becomes for $T_c = 2f_\pi$:

$$\frac{2m_q}{2f_\pi} = g = \frac{2\pi}{\sqrt{3}} \approx 3.628. \tag{30}$$

(The numerical near-agreement between 3.528 and 3.628 is no accident. This is because both equations require $E \sim p$: Eq.(29) due to very low energy acoustical (as opposed to optical phonon energy $p^2/2m$) phonons and Eq.(30) due to $E^2 = p^2 + m_\pi^2$ with $m_\pi = 0$ in the chiral limit).

The acoustical $ee$ Cooper pair is now replaced by a QCD $qq$ diquark in the "superconductive" region of Fig. 1, ie. for low $T$ and $\mu \approx \mu_c$. In general, the QCD energy gap is model dependent with superconductivity temperature

$$T_c^{(SC)} \approx \Delta/(\pi e^{-\gamma_E}) \approx 0.567\Delta. \tag{31}$$

Coloured diquarks suggest $T_c^{(SC)} < 180\ MeV$ [2]. This is of the order of $10^{12}\ ^oK$. Recall that low temperature superconductivity occurs at roughly $2\ ^oK$.

# 5 Conclusion

In Sec.2 we developed a chiral phase transition temperature $T_c = 2f_\pi^{CL} \approx 180\ MeV$ by independently melting the constituent quark mass, the scalar $\sigma$ mass and the quark condensate. In Sec.3 we noted that the above thermalization procedure leads uniquely to the SU(2) quark-level $L\sigma M$ field theory at $T = 0$. Then in Sec.4 we extended this picture (as shown in Fig. 1) to finite chemical potential.

With hindsight our approach to a thermal chiral field theory based on the quark-level $L\sigma M$ is to first work at $T = 0$ as summarized in Sec.3 and then proceed on to $T_c = 2f_\pi^{CL}$ as discussed in Sec.2. The reverse approach to a realistic low energy theory has also been studied [19]. Specifically one starts at a chiral restoration temperature $T_c$ (with $m_q(T_c) = 0$) involving only chiral bosons $\boldsymbol{\pi}$ and $\sigma$ [19] at $T = T_c \approx 200$ MeV. Then only a $\lambda \approx 20$ scale can generate a $L\sigma M$ field theory [20]. In our scheme, $g \approx 2\pi/\sqrt{3} \approx 3.6276$ and $\lambda = 2g^2 = 8\pi^2/3 \approx 26.3$, so it should not be surprising that $T_c = 2f_\pi^{CL} \approx 180$ MeV is near 200 MeV while $\lambda = 8\pi^2/3$ is near 20.

We propose that the above quark-level SU(2) $L\sigma M$ field theory generates a self-consistent thermal chiral phase transition with $T_c \approx 180\ MeV$ when $\mu = 0$, and $\mu_c \approx 325\ MeV$ when $T = 0$, and the superconductivity temperature is $T_c^{(SC)} \approx \Delta e^{\gamma_E}/\pi \approx 0.567\Delta$ for a QCD $qq$ energy gap $\Delta$. Coloured diquarks require $T_c^{(SC)} < 180\ MeV$ [2], as Fig. 1 suggests.

## Acknowledgments

M. D. S. appreciates hospitality of the Institute of Nuclear Physics in Kraków.

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Fig. 1. Schematic phase diagram